Solution to Exercise 2

1. Let $C = \left\{x \in \mathbb{R}^N : x_1 + x_2 t + x_3 t^2 + \cdots + x_N t^{N-1} \geq 0, \forall t \in [0,1]\right\}$, which is obviously a cone by definition.

$\forall \boldsymbol{x} = (x_1, \cdots, x_N) \in C$, $\boldsymbol{y} = (y_1, \cdots, y_N) \in C$ and $\lambda \in [0,1]$, we have

$$(\lambda x_1 + (1-\lambda) y_1) + \cdots + (\lambda x_N + (1-\lambda) y_N) t^{N-1} = \lambda (x_1 + \cdots + x_N t^{N-1}) + (1-\lambda)(y_1 + \cdots + y_N t^{N-1}) \geq 0.$$

for any $t \in [0,1]$, which implies C is convex.

Suppose $\left\{\boldsymbol{x}^{(n)}\right\}_{n=1}^{\infty} \subset C$ and $\lim_{n\to\infty} \boldsymbol{x}^{(n)} = \boldsymbol{x}^* = (x_1^*, x_2^*, \cdots, x_N^*)$. For fixed $t \in [0,1]$, we have $\lim_{n\to\infty} f_n(t) = x_1^* + x_2^* t + x_3^* t^2 + \cdots + x_N^* t^{N-1}$, where $f_n(t) = x_1^{(n)} + x_2^{(n)} t + \cdots + x_N^{(n)} t^{N-1}$. Then $\lim_{n\to\infty} f_n(t) \geq 0$, $\forall t \in [0,1]$, which implies C is closed.

To show C has non-empty interior, we note that $\boldsymbol{x}_0 = (1, \cdots, 1) \in \operatorname{int}(C)$. In fact, $\mathbb{B}(\boldsymbol{x}_0, \frac{1}{2}) \subset C$.

It remains to show C does not contain an entire line. Just note that $\forall$ non-zero $\boldsymbol{x} = (x_1, x_2, \cdots, x_N) \in C$, we must have $-\boldsymbol{x} \notin C$.

2. Suppose S is affine. We first assume $0 \in S$. Let $x \in S$ and $\gamma \in \mathbb{R}$. Since $0 \in S$, we have $\gamma x + (1-\gamma) 0 = \gamma x \in S$. Now, suppose $x, y \in S$. Then $x + y = 2\left(\frac{1}{2}x + \frac{1}{2}y\right) \in S$. Hence, S is closed under addition and scalar multiplication. Therefore, $S = 0 + S$ is a linear subspace. If $0 \notin S$, then $0 \in S - x$ for any $x \in S$. So $S - x$ is a linear subspace. Therefore, $S = x + V$. The other direction is simple, just use the fact that V is a linear subspace.

3. Let V be the subspace parallel to S. Then $S - x_0 = V$. Hence $\operatorname{span}\{x_1 - x_0, \ldots, x_m - x_0\} \subseteq V$. Let $x \in V$, then $x + x_0 \in S$. So

$$x + x_0 = \sum_{i=0}^{m} \lambda_i x_i, \text{ where } \sum \lambda_i = 1$$

Therefore

$$x = \sum_{i=1}^{m} \lambda_i (x_i - x_0) \in \operatorname{span}\{x_1 - x_0, x_m - x_0\}$$